ARO 26394.5-MA

AD-A238 538

(2)

THE ASYMPTOTIC DISTRIBUTIONS OF AUTOREGRESSIVE COEFFICIENTS

DTIC
ELECTE
JUL 22 1991
S C D

T. W. Anderson
Stanford University

TECHNICAL REPORT NO. 26
APRIL 1991

U. S. Army Research Office
Contract DAAL03-89-K-0033
Theodore W. Anderson, Project Director

Department of Statistics
Stanford University
Stanford, California

Approved for Public Release; Distribution Unlimited.

91-05301

91 7 17 026

THE ASYMPTOTIC DISTRIBUTION OF AUTOREGRESSIVE COEFFICIENTS

T. W. Anderson

Stanford University

Accession For	
NTIS GRA&I	☑
DTIC TAB	☐
Unannounced	☐
Justification	
By	
Distribution/	
Availability Codes	
Dist	Avail and/or Special
A-1	

TECHNICAL REPORT NO. 26

APRIL 1991

U. S. Army Research Office

Contract DAAL03-89-K-0033

Theodore W. Anderson, Project Director

Department of Statistics

Stanford University

Stanford, California

Approved for Public Release; Distribution Unlimited.

1. Introduction. A stationary stochastic process $\{x_t\}$ with finite second moment has mean $\mathcal{E}x_t = \mu$ and covariance or autocovariance sequence

$$(1.1) \qquad \mathcal{E}(x_t - \mu)(x_{t+s} - \mu) = \sigma(s), \qquad s = 0, \pm 1, \dots .$$

The covariance sequence can be expressed in terms of the variance of the process $\sigma(0)$ and the correlation or autocorrelation sequence

$$(1.2) \qquad \rho_s = \frac{\sigma(s)}{\sigma(0)}, \qquad s = 0, \pm 1, \dots .$$

If the process is Gaussian, the mean and the covariance sequence or alternatively the mean, variance, and correlation sequence completely specify the process.

Inference about the process may be based on a sample of n consecutive observations $x_1, \dots, x_n$. The sample covariance sequence may be defined as

$$(1.3) \qquad c_h = c_{-h} = \frac{1}{n}\sum_{t=1}^{n-h}(x_t - \mu)(x_{t+h} - \mu), \qquad h = 0, 1, \dots, n-1,$$

if μ is known and as

$$(1.4) \qquad c_h^* = c_{-h}^* = \frac{1}{n}\sum_{t=1}^{n-h}(x_t - \bar{x})(x_{t+h} - \bar{x}), \qquad h = 0, 1, \dots, n-1,$$

if μ is unknown; here $\bar{x} = \sum_{t=1}^{n} x_t/n$. The sample correlation sequence is defined as $r_h = c_h/c_0$ or as $r_h^* = c_h^*/c_0^*$, $h = 0, \pm 1, \dots, \pm(n-1)$, in the two cases. If the process is Gaussian, the sample mean and covariance sequence or alternatively the sample mean, variance, and correlation sequence constitute a sufficient set of statistics. In any case the sample correlations may be used for making inferences about the pattern of dependence in the process.

Under general conditions on the process, any finite set of correlations is asymptotically normally distributed as $n \to \infty$. More precisely, the limiting distribution

of $\sqrt{n}(r_1 - \rho_1), \ldots, \sqrt{n}(r_H - \rho_H)$ for arbitrary H is normal with mean 0 and a covariance matrix, say, W. If x_t is a linear process

$$x_t = \mu + \sum_{i=0}^{\infty} \gamma_i v_{t-i}, \qquad t = 0, \pm 1, \ldots, \tag{1.5}$$

with $\sum_{i=0}^{\infty} \gamma_i^2 < \infty$,

$$\sum_{i=0}^{\infty} |\gamma_i| < \infty \tag{1.6}$$

and $\{v_t\}$ a sequence of independently identically distributed random variables with $\mathcal{E} v_t = 0$ and $\mathcal{E} v_t^2 = \sigma^2$, the elements of W are

$$w_{gh} = \sum_{r=1}^{\infty} (\rho_{r+g} + \rho_{r-g} - 2\rho_r \rho_g)(\rho_{r+h} + \rho_{r-h} - 2\rho_r \rho_h), \qquad g, h = 1, \ldots, H. \tag{1.7}$$

The covariance and correlation sequences of $\{x_t\}$ defined by (1.5) are

$$\sigma(h) = \sigma(-h) = \sigma^2 \sum_{i=0}^{\infty} \gamma_i \gamma_{i+h}, \qquad h = 0, 1, \ldots, \tag{1.8}$$

$$\rho_h = \rho_{-h} = \frac{\sum_{i=0}^{\infty} \gamma_i \gamma_{i+h}}{\sum_{i=0}^{\infty} \gamma_i^2}, \qquad h = 0, 1, \ldots . \tag{1.9}$$

If $\{x_t\}$ is second-order stationary and purely nondeterministic, it can be represented as (1.5) with the v_t's having mean 0 and variance σ^2 and being uncorrelated. In this situation the v_t's are the innovations and the errors in prediction one step ahead.

A type of process that has been particularly useful is the autoregressive process which satisfies the stochastic difference equation

$$\sum_{j=0}^{p} \beta_j (x_{t-j} - \mu) = v_t, \qquad t = \ldots, -1, 0, 1, \ldots, \tag{1.10}$$

where $\beta_0 = 1$. If the v_t's are independently identically distributed and the roots of

$$(1.11) \qquad \sum_{i=0}^{p} \beta_i z^{p-i} = 0$$

are less than 1 in absolute value, (1.10) defines a stationary process. If $\mathcal{E} v_t = 0$ and $\mathcal{E} v_t^2 = \sigma^2 < \infty, \{x_t\}$ has a representation (1.5), $\mathcal{E} x_t = \mu$, and the covariance and correlation sequences are given by (1.8) and (1.9), respectively. In the case of $p = 1$ $\rho_1 = -\beta_1$ and r_1 or r_1^* is an estimator of $-\beta_1$.

A *serial correlation coefficient* (of order one) is r_1 or r_1^* or a close approximation to r_1 or r_1^*. It is of particular interest as providing a test of the null hypothesis that a sequence of observations is independently distributed. The alternative hypothesis is (often unstated) that the observations come from an autoregressive process of order 1. The doctoral dissertation of Geoffrey Watson (1952) dealt with serial correlations; hence this present paper is related to the early work of Watson.

In the early '40's at least four people nearly simultaneously and essentially independently derived the distribution of some form of serial correlation when the x_t's were independently normally distributed with mean 0 and variance σ^2 ($\mu = 0$ and $\beta_1 = 0$). John von Neumann (1941) treated

$$(1.12) \qquad \begin{aligned} &\frac{\sum_{t=2}^{n}(x_t - x_{t-1})^2}{\sum_{t=1}^{n}(x_t - \bar{x})^2} \\ &\qquad = 2 - 2\frac{\sum_{t=2}^{n}(x_t - \bar{x})(x_{t-1} - \bar{x}) + \frac{1}{2}(x_1 - \bar{x})^2 + \frac{1}{2}(x_n - \bar{x})^2}{\sum_{t=1}^{n}(x_t - \bar{x})^2}. \end{aligned}$$

The fraction on the right-hand side of (1.12) is a serial correlation coefficient. Tjalling Koopmans (1942) studied the distribution of r_1 and approximations to the distribution. R. L. Anderson (1942) found the distribution of the circular form (suggested by Harold Hotelling)

$$(1.13) \qquad \frac{\sum_{t=1}^{n}(x_t - \bar{x})(x_{t-1} - \bar{x})}{\sum_{t=1}^{n}(x_t - \bar{x})^2},$$

where $x_0 \equiv x_n$. Wilfrid J. Dixon had independently obtained many of the results. (They were to be included in his doctoral dissertation, but because of the publications of the others he changed his dissertation topic.) A number of new results were

published subsequently [Dixon (1944)]. These authors also derived some asymptotic distributions of the serial correlations, which are asymptotically equivalent to r_1.

R. L. Anderson and T. W. Anderson (1950) observed that the distribution of the circular serial correlation of residuals from a fitted Fourier series has the same form as the circular serial correlation from the mean. T. W. Anderson (1948) generalized this result to any serial correlation of residuals from independent variables constituting characteristic vectors of the matrix of the quadratic form in the numerator of the coefficient. R. L. Anderson suggested "to one of the authors" of Durbin and Watson (1950), (1951) the investigation of more general residuals. The dissertation of Watson (1952), a more extensive study of serial correlation of residuals, was under the supervision of R. L. Anderson.

Mann and Wald (1943) derived the asymptotic normality of the sample autocovariances in an autoregressive process under the assumption of existence of all moments of the innovations. The asymptotic covariances (1.7) of the sample correlations for the linear process were given by Bartlett (1946) under the (implicit) assumption that $\mathrm{E}v_t^4 < \infty$. Hoeffding and Robbins (1948) proved the asymptotic normality under the assumption $\mathcal{E}v_t^6 < \infty$ for (1.5) being a finite sum; this condition was weakened by Diananda (1953) and Walker (1954) to $\mathcal{E}v_t^4 < \infty$. The fact that the asymptotic covariances depend only on the process correlation function suggests that the condition $\mathcal{E}v_t^4 < \infty$ is not necessary. T. W. Anderson (1959) obtained the limit distribution of $\sqrt{n}(r_1 - \rho_1)$ for an autoregressive process of order one when only $\mathcal{E}v_t^2 < \infty$ is assumed. T. W. Anderson and Walker (1964) found the asymptotic distribution of a finite number of correlations when (1.5) is satisfied with $\sum_{i=0}^{\infty} i\gamma_i^2 < \infty$.

The assumption that the process $\{x_t\}$ is stationary can be relaxed; in the model (1.5) the v_t's do not have to be identically distributed. In that case the process correlations can still be defined by (1.9). Moreover, the innovations do not need to be independently distributed. Instead we assume that the v_t's are martingale differences. Let $\ldots \subset \mathcal{F}_{-1} \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots$ be an increasing sequence of σ-fields such that v_t is $\mathcal{F}_t$-measurable. We assume that

$$\mathcal{E}(v_t|\mathcal{F}_{t-1}) = 0 \text{ a.s.}, \qquad t = \ldots, -1, 0, 1, \ldots . \tag{1.14}$$

Instead of $\mathcal{E}v_t^2 = \sigma^2$ as for identically distributed random variables, we assume

$$\mathcal{E}(v_t^2|\mathcal{F}_{t-1}) = \sigma_t^2 \text{ a.s.}, \qquad t = -1, 0, 1, \ldots, \tag{1.15}$$

$$\frac{1}{n}\sum_{t=1}^{n} \sigma_t^2 \xrightarrow{\text{P}} \sigma^2 > 0, \tag{1.16}$$

and

$$\sup_{t=1,2,\ldots} \mathcal{E}[v_t^2 I(v_t^2 > a)|\mathcal{F}_{t-1}] \xrightarrow{\text{P}} 0 \tag{1.17}$$

as $a \to \infty$; here $I(\cdot)$ is the indicator function. [$I(A) = 1$ if the event A occurs, and $I(A) = 0$ if the event A does not occur.] Condition (1.17) is a kind of conditional uniform integrability. For convenience we assume that for some K

$$\mathcal{E}v_t^2 \leq K, \qquad t = \ldots, -1, 0, -1, \ldots . \tag{1.18}$$

$$\mathcal{E}v_t^2 v_s^2 \leq K^2, \qquad t \neq s. \tag{1.19}$$

Finally, to define the asymptotic second-order moments of r_h we need more information on the mixed fourth-order moments of the v_t's. We assume

$$\frac{1}{n}\sum_{t=1}^{n} \sigma_t^2 v_{t-r} v_{t-s} \xrightarrow{\text{P}} \delta_{rs}\sigma^4 \qquad r, s = 1, 2, \ldots, \tag{1.20}$$

where $\delta_{ss} = 1$ and $\delta_{rs} = 0$, $r \neq s$. In addition there is the technical condition

$$\sum_{i=0}^{\infty} \sqrt{i}\gamma_i^2 < \infty. \tag{1.21}$$

Note that autoregressive moving average processes satisfy (1.21). Hannan and Heyde (1972) derived the asymptotic distribution of the r_j's when the v_t's are

martingale differences. Their conditions are much stronger than ours. In particular, they assume $\sigma_t^2 = \sigma^2$ a.s. and bounded fourth-order moments. A more detailed comparison of their conditions and ours will be made later in this paper.

The purpose of this paper is to give the very weak conditions for the asymptotic normal distribution of the sample correlations. A general theorem of Anderson and Kunitomo (1989) is used. That theorem is based on a martingale central limit theorem of Dvoretzky (1972). More details are given in this paper than in Hannan and Heyde (1972).

The organization of this paper follows that of Anderson and Walker (1964). The case of one sample correlation when μ is known is proved first. Then the theorem is proved for an arbitrary number of correlations. Finally the result is proved for unknown μ.

Theorem. Let $\{x_t\}$ be defined by (1.5), where $\{\gamma_i\}$ satisfies (1.6) and (1.21). Suppose that $\{v_t\}$ satisfies (1.14), (1.15), (1.16), (1.17), (1.18), (1.19), and (1.20). Then

$$\sqrt{n}\begin{bmatrix} r_1 - \rho_1 \\ \vdots \\ r_H - \rho_H \end{bmatrix} \xrightarrow{\mathrm{d}} N(0, W), \tag{1.22}$$

where $W = (w_{gh})$ and w_{gh} is given by (1.7).

Corollary. Under the conditions of the theorem

$$\sqrt{n}\begin{bmatrix} r_1^* - \rho_1 \\ \vdots \\ r_H^* - \rho_H \end{bmatrix} \xrightarrow{\mathrm{d}} N(0, W), \tag{1.23}$$

where $W = (w_{gh})$ and w_{gh} is given by (1.7).

2. Proof of Theorem. The proof follows the pattern of Anderson and Walker (1964). [See also Hannan and Heyde (1972).] First we prove that $\sqrt{n}(r_l - \rho_l) \xrightarrow{\mathrm{d}} N(0, w_{ll})$; then we take up the case of an arbitrary number of correlations. Let $y_t = x_t - \mu$. Define

$$z_n^{(l)} = \frac{1}{\sqrt{n}} \left(\sum_{t=1}^{n-l} y_t y_{t+l} - \rho_l \sum_{t=1}^{n} y_t^2 \right), \tag{2.1}$$

$$y_{tk} = \sum_{i=0}^{k} \gamma_i v_{t-i}, \tag{2.2}$$

$$\rho_{lk} = \frac{\sum_{i=0}^{k-l} \gamma_i \gamma_{i+l}}{\sum_{i=0}^{n} \gamma_i^2}, \tag{2.3}$$

$$\begin{aligned} z_{nk}^{(l)} &= \frac{1}{\sqrt{n}} \left(\sum_{t=1}^{n-l} y_{tk} y_{t+l,k} - \rho_{lk} \sum_{t=1}^{n} y_{tk}^2 \right) \\ &= \frac{1}{\sqrt{n}} \left(\sum_{t=1}^{n-l} \sum_{i,j=0}^{k} \gamma_i \gamma_j v_{t-i} v_{t+l-j} - \rho_{lk} \sum_{t=1}^{n} \sum_{i,j=0}^{k} \gamma_i \gamma_j v_{t-i} v_{t-j} \right) \\ &= \frac{1}{\sqrt{n}} \left(\sum_{t=1}^{n-l} \sum_{i=0}^{k} \sum_{h=-l}^{k-l} \gamma_i \gamma_{h+l} v_{t-i} v_{t-h} - \rho_{kl} \sum_{t=1}^{n} \sum_{i,j=0}^{k} \gamma_i \gamma_j v_{t-i} v_{t-j} \right), \end{aligned} \tag{2.4}$$

$$z_{nk}^{(l)*} = \frac{1}{\sqrt{n}} \left(\sum_{t=1}^{n-l} \sum_{i=0}^{k} \sum_{\substack{h=-l \\ h \neq i}}^{k-l} \gamma_i \gamma_{h+l} v_{t-i} v_{t-h} - \rho_{lk} \sum_{t=1}^{n} \sum_{\substack{i,j=0 \\ i \neq j}}^{k} \gamma_i \gamma_j v_{t-i} v_{t-j} \right). \tag{2.5}$$

Note that terms in the summation in (2.5) with $t-i=t-h$ and $t-i=t-j$ are omitted.

Lemma 1. The limiting distribution of $z_{n,k}^{(l)*}$ as $n \to \infty$ is $N(0, \sigma^4 V_{lk})$ where

$$V_{lk} = \sum_{r=1}^{k+l} \left[\delta_{rk}^{(l)} \right]^2, \tag{2.6}$$

$$\delta_{rk}^{(l)} = \sum_{i=0}^{k} [\gamma'_i \gamma'_{i+l+r} + \gamma'_i \gamma'_{i+l-r} - \rho_{lk}(\gamma'_i \gamma'_{i+r} + \gamma'_i \gamma'_{i-r})] \tag{2.7}$$

and $\gamma'_i = \gamma_i$ for $0 \le i \le k$ and $\gamma'_i = 0$ otherwise (that is, for $i < 0$ and $i > k$).

Proof of Lemma 1. By changing indices of summation we can write

$$(2.8)\quad z_{nk}^{(l)*} = \frac{1}{\sqrt{n}}\Bigg[\sum_{t=1}^{n-l}\sum_{i=0}^{k}\left(\sum_{r=1}^{k}\gamma'_i\gamma'_{i+l+r}v_{t-i}v_{t-i-r} + \sum_{r=1}^{k+l}\gamma'_i\gamma'_{i+l-r}v_{t-i}v_{t-i+r}\right)$$
$$- \rho_{lk}\sum_{t=1}^{n}\sum_{i=0}^{k}\left(\sum_{r=1}^{k}\gamma'_i\gamma'_{i+r}v_{t-i}v_{t-i-r} + \sum_{r=1}^{k}\gamma'_i\gamma'_{i-r}v_{t-i}v_{t-i+r}\right)\Bigg].$$

Another change of indices and the addition and subtraction of a finite number of terms (the number not depending on n) changes (2.8) to

$$(2.9)\qquad \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\sum_{r=1}^{k+l}\delta_{rk}^{(l)}v_t v_{t-r} = \frac{1}{\sqrt{n}}\sum_{t=1}^{n}x_{tk}^{(l)}v_t,$$

where $x_{tk}^{(l)} = \sum_{r=1}^{k+l}\delta_{rk}^{(l)}v_{t-r}$. We show that (2.9) has the limiting distribution $N(0, \sigma^2 V_{lk})$ by means of Theorem 1 of Anderson and Kunitomo (1989). Condition (1.17) implies

$$(2.10)\qquad \sup_{t=-k+1,\ldots,n}\frac{v_t^2}{n} \xrightarrow{\mathrm{p}} 0.$$

[See the proof of Theorem 2.23 of Hall and Heyde (1980).] Therefore

$$(2.11)\qquad \sup_{t=1,\ldots,n}\frac{|x_{tk}^{(l)}|^2}{n} \xrightarrow{\mathrm{p}} 0.$$

Furthermore (1.16) and (1.17) imply

$$(2.12)\qquad \frac{1}{n}\sum_{t=1}^{n}v_{t-r}^2 \xrightarrow{\mathrm{p}} \sigma^2.$$

[See Anderson and Kunitomo (1989), Theorem 2, for example.] Then

$$(2.13) \qquad \frac{1}{n}\sum_{t=1}^{n} v_{t-r}v_{t-s} \xrightarrow{\mathrm{P}} 0, \qquad r \neq s.$$

[Application of Theorem 1 of Anderson and Kunitomo (1989), for example, with v_{t-s} replaced by v_t and v_{t-r} replaced by z_t for $s < r$ shows that $\sqrt{n}$ times the left-hand side of (2.13) has a limiting normal distribution.] Thus

$$(2.14) \qquad \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left[x_{tk}^{(l)}\right]^2 \xrightarrow{\mathrm{P}} \sum_{r=1}^{k+l}\left[\delta_{rk}^{(l)*}\right]^2 = V_{lk}.$$

The conditions of Theorem 1 of Anderson and Kunitomo (1989) are satisfied with $z_t = x_{tk}^{(l)}$. This completes the proof of Lemma 1. ∎

Let

$$(2.15) \qquad z_n^{(l)*} = \frac{1}{\sqrt{n}}\left(\sum_{t=1}^{n-l}\sum_{\substack{i,j=0 \\ i\neq j-l}}^{\infty}\gamma_i\gamma_j v_{t-i}v_{t+l-j} - \rho_l\sum_{t=1}^{n}\sum_{\substack{i,j=0 \\ i\neq j}}^{\infty}\gamma_i\gamma_j v_{t-i}v_{t-j}\right).$$

Lemma 2. The limiting distribution of $z_n^{(l)*}$ as $n \to \infty$ is $N(0, \sigma^4 V_l)$, where

$$(2.16) \qquad V_l = \left(\sum_{i=0}^{\infty}\gamma_i^2\right)^2 w_{ll}.$$

Proof of Lemma 2. The limit of $N(0, \sigma^4 V_{lk})$ is $N(0, \sigma^4 V_l)$ as $k \to \infty$ since

$$(2.17) \qquad V_{lk} \to \frac{1}{2}\sum_{r=1}^{\infty}\left\{\sum_{i=0}^{\infty}[\gamma_i^*\gamma_{i+l+r}^* + \gamma_i^*\gamma_{i+l-r}^* - \rho_l(\gamma_i^*\gamma_{i+r}^* + \gamma_i^*\gamma_{i-r}^*)]\right\}^2 = V_l,$$

where $\gamma_i^* = \gamma_i$ for $0 \leq i$ and $\gamma_i^* = 0$ for $i < 0$. To complete the proof of Lemma 2 we shall show that $z_n^{(l)*} - z_{nk}^{(l)*} \xrightarrow{\mathrm{P}} 0$ as $k \to \infty$ uniformly in T. Let

$$u_{tk} = y_t - y_{tk} = \sum_{t=k+1}^{\infty} \gamma_i v_{t-i}. \tag{2.18}$$

Then

$$\begin{aligned} z_n^{(l)*} - z_{nk}^{(l)*} &= \frac{1}{\sqrt{n}} {\sum_{t=1}^{n-l}}' [(y_{tk} + u_{tk})(y_{t+l,k} + u_{t+l,k}) - y_{tk} y_{t+l,k}] \\ &\quad + \frac{1}{\sqrt{n}} {\sum_{t=1}^{n}}' [-\rho_l (y_{tk} + u_{tk})^2 + \rho_{lk} y_{tk}^2] \\ &= \frac{1}{\sqrt{n}} {\sum_{t=1}^{n-l}}' [u_{tk} y_{t+l,k} + y_{t,k} u_{t+l,k} + u_{tk} u_{t+l,k}] \\ &\quad + \frac{1}{\sqrt{n}} {\sum_{t=1}^{n}}' [(\rho_{lk} - \rho_l) y_{tk}^2 - 2\rho_l y_{tk} u_{tk} - \rho_l u_{tk}^2], \end{aligned} \tag{2.19}$$

where the prime on $\sum'$ denotes that terms $v_t v_s$ with $t = s$ are omitted. Let

$$T_1 = {\sum_{t=1}^{n-l}}' u_{tk} y_{t+l,k}, \ldots, T_6 = -\rho_l {\sum_{t=1}^{n}}' u_{tk}^2. \tag{2.20}$$

Then (2.19) is $(1/\sqrt{n}) \sum_{h=1}^{6} T_h$.

Consider, for example,

$$\begin{aligned} -\frac{T_6}{\rho_l} &= {\sum_{t=1}^{n}}' u_{tk}^2 \\ &= 2 \sum_{t=1}^{n} \sum_{\substack{i,j=k+1 \\ i<j}}^{\infty} \gamma_i \gamma_j v_{t-i} v_{t-j}. \end{aligned} \tag{2.21}$$

(If $\rho_l = 0$, $T_4 = T_5 = T_6 = 0$.) The expected value of the square of (2.21) divided by 2 is

$$\begin{aligned} &\mathcal{E} \left(\frac{1}{\sqrt{n}} \sum_{t=1}^{n} \sum_{\substack{i,j=k+1 \\ i<j}}^{\infty} \gamma_i \gamma_j v_{t-i} v_{t-j} \right)^2 \\ &= \frac{1}{n} \mathcal{E} \sum_{t,t'=1}^{n} \sum_{\substack{i,j,i',j'=k+1 \\ i<j, i'<j'}}^{\infty} \gamma_i \gamma_j \gamma_{i'} \gamma_{j'} v_{t-i} v_{t-j} v_{t'-i'} v_{t'-j'}. \end{aligned} \tag{2.22}$$

By the Cauchy-Schwarz inequality

$$\begin{aligned}(2.23)\qquad \mathcal{E}|v_{t-i}v_{t-j}v_{t'-i'}v_{t'-j'}| &\le \sqrt{\mathcal{E}v_{t-i}^2v_{t-j}^2\mathcal{E}v_{t'-i'}^2v_{t'-j'}^2} \\ &\le K^2.\end{aligned}$$

Then (2.22) is

$$(2.24)\qquad \frac{1}{n}\sum_{\substack{i,j,i',j'=k+1\\ i<j,i'<j'}}^{\infty}\gamma_i\gamma_j\gamma_{i'}\gamma_{j'}\sum_{t=1}^{n}\mathcal{E}v_{t-j}v_{t-i-(j'-i')}\mathcal{E}(v_{t-i}^2|\mathcal{F}_{t-i-1}),$$

where the sum on t is vacuous unless $1 \le t-i+i' \le n$. Use of (2.23) shows that (2.24) is not greater than

$$(2.25)\qquad \left(\sum_{i=k+1}^{\infty}|\gamma_i|\right)^4 K^2.$$

This quantity is made arbitrarily small by taking k sufficiently large. Hence $T_6 \xrightarrow{\text{p}} 0$ uniformly in n as $k \to \infty$.

For T_4 we consider

$$\begin{aligned}(2.26)\qquad \frac{1}{\sqrt{n}}{\sum}' y_{tk}^2 &= \frac{2}{\sqrt{n}}\sum_{t=1}^{n}\sum_{\substack{i,j=0\\ i<j}}^{k}\gamma_i\gamma_j v_{t-i}v_{t-j} \\ &= \frac{2}{\sqrt{n}}\sum_{t=1}^{n}\sum_{i=0}^{k-1}\sum_{j=i+1}^{k}\gamma_i\gamma_j v_{t-i}v_{t-j} \\ &= \frac{2}{\sqrt{n}}\sum_{t=1}^{n}\sum_{i=0}^{k-1}\sum_{h=1}^{k-i}\gamma_i\gamma_{i+h}v_{t-i}v_{t-i-h} \\ &= 2\sum_{i=0}^{k-1}\gamma_i\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left(\sum_{h=1}^{k-i}\gamma_{i+h}v_{t-i-h}\right)v_{t-i}.\end{aligned}$$

For each i the term in (2.26) that is multiplied by γ_i has a limiting normal distribution [by Theorem 1 of Anderson and Kunitomo (1989) with $z_t = \sum_{h=1}^{k-i}\gamma_{i+h}v_{t-i-h}$], and hence the sum on i has a limiting distribution. Since $\rho_{lk} \to \rho_l$, the term $T_4 \xrightarrow{\text{p}} 0$.

It follows from the Cauchy-Schwarz inequality and the behavior of $\sum' y_{tk}^2/\sqrt{n}$ and $\sum' u_{tk}^2/\sqrt{n}$ that $T_5 \xrightarrow{\text{P}} 0$. That the first three terms converge stochastically to 0 follows similarly. ∎

Lemma 3. The limiting distribution of $z_n^{(l)}$ is $N(0, \sigma^4 V_l)$.

Proof of Lemma 3. We have

$$
\begin{aligned}
(2.27) \qquad z_n^{(l)} - z_n^{(l)*} &= \frac{1}{\sqrt{n}} \left[\sum_{t=1}^{n-l} \sum_{i=0}^{\infty} \gamma_i \gamma_{i+l} v_{t-i}^2 - \rho_l \sum_{t=1}^{n} \sum_{i=0}^{\infty} \gamma_i^2 v_{t-i}^2 \right] \\
&= \frac{1}{\sqrt{n}} \left[\sum_{i=0}^{\infty} \gamma_i \gamma_{i+l} \sum_{s=1-i}^{n-l-i} v_s^2 - \rho_l \sum_{i=0}^{\infty} \gamma_i^2 \sum_{s=1-i}^{n-i} v_s^2 \right] \\
&= \frac{1}{\sqrt{n}} \left[\sum_{i=0}^{\infty} \gamma_i \gamma_{i+l} T_{ni}^{(l)} - \rho_l \sum_{i=0}^{\infty} \gamma_i^2 T_{ni}^{(0)} \right],
\end{aligned}
$$

where

$$
\begin{aligned}
(2.28) \qquad T_{ni}^{(l)} &= \sum_{s=1-i}^{n-l-i} v_s^2 - \sum_{s=1}^{n} v_s^2, \\
&= \sum_{s=1-i}^{0} v_s^2 - \sum_{s=n-l-i+1}^{n} v_s^2, \qquad i \leq n-l.
\end{aligned}
$$

$$
\begin{aligned}
(2.29) \qquad T_{ni}^{(0)} &= \sum_{s=1-i}^{n-i} v_s^2 - \sum_{s=1}^{n} v_s^2 \\
&= \sum_{s=1-i}^{0} v_s^2 - \sum_{s=n-i+1}^{n} v_s^2, \qquad i \leq n.
\end{aligned}
$$

The second term on the right-hand side of (2.27) is ρ_l times

$$
\begin{aligned}
(2.30) \qquad \frac{1}{\sqrt{n}} \sum_{i=0}^{\infty} \gamma_i^2 T_{ni}^{(0)} &= \frac{1}{\sqrt{n}} \left(\sum_{i=0}^{n} \gamma_i^2 \sum_{s=1-i}^{0} v_s^2 + \sum_{i=n+1}^{\infty} \gamma_i^2 \sum_{s=1-i}^{n-i} v_s^2 \right) \\
&\quad - \frac{1}{\sqrt{n}} \left(\sum_{i=0}^{n} \gamma_i^2 \sum_{s=n-i+1}^{n} v_s^2 + \sum_{i=n+1}^{\infty} \gamma_i^2 \sum_{s=1}^{n} v_s^2 \right).
\end{aligned}
$$

The expected value of the first pair of terms on the right-hand side of (2.30) is not greater than

$$(2.31)\quad \frac{1}{\sqrt{n}}\left(\sum_{i=0}^{n} i\gamma_i^2 + \sum_{i=n+1}^{\infty} n\gamma_i^2\right)K = \left(\sum_{i=0}^{n}\sqrt{\frac{i}{n}}\sqrt{i}\gamma_i^2 + \sum_{i=n+1}^{\infty}\sqrt{\frac{n}{i}}\sqrt{i}\gamma_i^2\right)K$$
$$\leq \left(\sum_{i=0}^{m}\sqrt{\frac{i}{n}}\sqrt{i}\gamma_i^2 + \sum_{i=m+1}^{\infty}\sqrt{i}\gamma_i^2\right)K$$
$$\leq \left(\sqrt{\frac{m}{n}}\sum_{i=0}^{\infty}\sqrt{i}\gamma_i^2 + \sum_{i=m+1}^{\infty}\sqrt{i}\gamma_i^2\right)K$$

for $m \leq n$. The second term on the right-hand side of (2.31) is made arbitrarily small by taking m large enough. The first term is made arbitrarily small by taking n large enough, given m. The second pair of terms on the right-hand side of (2.30) is treated similarly. Then Markov's inequality implies that (2.30) converges stochastically to 0.

The absolute value of the first term on the right-hand side of (2.27) has expected value not greater than

$$(2.32)\frac{1}{\sqrt{n}}\sum_{i=0}^{\infty}|\gamma_i|\cdot|\gamma_{i+l}|\cdot\mathcal{E}|T_{ni}^{(l)}|$$
$$\leq \frac{1}{\sqrt{n}}\left[\sum_{i=0}^{n-l}|\gamma_i|\cdot|\gamma_{i+l}|(2i+l) + \sum_{i=n-l+1}^{\infty}|\gamma_i|\cdot|\gamma_{i+l}|(2n-l)\right]K$$
$$\leq \frac{1}{\sqrt{n}}\left[2\sum_{i=0}^{n-l}|\gamma_i|\cdot|\gamma_{i+l}|i + 2\sum_{i=n-l+1}^{\infty}|\gamma_i|\cdot|\gamma_{i+l}|n + \sum_{i=0}^{\infty}|\gamma_i|\cdot|\gamma_{i+l}|l\right]K$$
$$\leq \frac{2}{\sqrt{n}}\left[\sum_{i=0}^{n}\gamma_i^2 i + n\sum_{i=n-l+1}^{\infty}\gamma_i^2\right]K + \frac{l}{\sqrt{n}}\sum_{i=0}^{\infty}\gamma_i^2 K.$$

The argument used for (2.31) shows that the right-hand side of (2.32) can be made arbitrarily small uniformly in n. ∎

Lemma 4. Under the assumptions of the theorem

$$(2.33)\qquad c_0 \xrightarrow{P} \sigma^2\sum_{i=0}^{\infty}\gamma_i^2.$$

Proof of Lemma 4. We can write c_0 as

$$(2.34)\qquad \frac{1}{n}\sum_{t=1}^{n} y_t^2 = \frac{1}{n}\sum_{t=1}^{n} y_{tk}^2 + \frac{2}{n}\sum_{t=1}^{n} y_{tk}u_{tk} + \frac{1}{n}\sum_{t=1}^{n} u_{tk}^2.$$

The last term in (2.34), which is nonnegative, has expected value

$$\begin{aligned}(2.35)\qquad \mathcal{E}\frac{1}{n}\sum_{t=1}^{n} u_{tk}^2 &= \mathcal{E}\frac{1}{n}\sum_{t-1}^{n}\sum_{i,j=k+1}^{\infty} \gamma_i\gamma_j v_{t-i}v_{t-j} \\ &\le \frac{1}{n}\sum_{t=1}^{n}\sum_{i,j=k+1}^{\infty} |\gamma_i||\gamma_j|\mathcal{E}|v_{t-i}v_{t-j}| \\ &\le \left(\sum_{i=k+1}^{\infty}|\gamma_i|\right)^2 K.\end{aligned}$$

This can be made arbitrarily small by taking k sufficiently large.

The first term of (2.34) is

$$\begin{aligned}(2.36)\qquad \frac{1}{n}\sum_{t=1}^{n} y_{tk}^2 &= \frac{1}{n}\sum_{t=1}^{n}\sum_{i,j=0}^{k}\gamma_i\gamma_j v_{t-i}v_{t-j} \\ &= \sum_{i=0}^{k}\gamma_i^2\frac{1}{n}\sum_{t=1}^{n} v_{t-i}^2 + \frac{1}{n}\sum_{t=1}^{n}\sum_{\substack{i,j=0\\ i\neq j}}^{k}\gamma_i\gamma_j v_{t-i}v_{t-j}.\end{aligned}$$

The first term on the right-hand side of (2.36) converges in probability to $\sigma^2\sum_{i=0}^{k}\gamma_i^2$, the limit of which is $\sigma^2\sum_{i=0}^{\infty}\gamma_i^2$ as $k \to \infty$. The second term in (2.36) is $2/\sqrt{n}$ times

$$(2.37)\qquad \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\sum_{j=0}^{k-1}\sum_{i=j+1}^{k}\gamma_i\gamma_j v_{t-i}v_{t-j} = \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\sum_{j=0}^{k-1}\sum_{h=1}^{k-j}\gamma_j\gamma_{j+h}v_{t-j}v_{t-j-h}.$$

Addition and subtraction of $k(k-1)$ terms changes (2.37) into

$$(2.38)\qquad \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left(\sum_{j=0}^{k-1}\sum_{h=1}^{k-j}\gamma_j\gamma_{j+h}v_{t-h}\right)v_t.$$

Theorem 1 of Anderson and Kunitomo (1989) with $z_t = \sum_{j=0}^{k-1} \sum_{h=1}^{k-j} \gamma_j \gamma_{j+h} v_{t-h}$ shows that the limiting distribution of (2.38) is normal with finite variance. Hence the second term in (2.36) converges stochastically to 0.

The second term in (2.34) is in absolute value

$$2\left|\frac{1}{n}\sum_{t=1}^{n} y_{tk}u_{tk}\right| \leq 2\sqrt{\frac{1}{n}\sum_{t=1}^{n} y_{tk}^2}\sqrt{\frac{1}{n}\sum_{t=1}^{n} u_{tk}^2}, \tag{2.39}$$

which converges stochastically to 0 by the preceding results. The lemma follows. ∎

Lemma 5. Under the conditions of the theorem $\sqrt{n}(r_l - \rho_l) \xrightarrow{\mathrm{d}} N(0, w_{ll})$.

Proof of Lemma 5. We have

$$\begin{aligned} \sqrt{n}(r_l - \rho_l) &= \sqrt{n}\left[\frac{\sum_{t=1}^{n-l} y_t y_{t+l}}{\sum_{t=1}^{n} y_t^2} - \rho_l\right] \\ &= \frac{\frac{1}{\sqrt{n}}\left[\sum_{t=1}^{n-l} y_t y_{t+l} - \rho_l \sum_{t=1}^{n} y_t^2\right]}{\frac{1}{n}\sum_{t=1}^{n} y_t^2}. \end{aligned} \tag{2.40}$$

Lemma 5 follows from Lemmas 1 to 4. ∎

The theorem is proved by showing that

$$\begin{aligned} \sqrt{n}\sum_{h=1}^{H} \alpha_h(r_h - \rho_h) &= \frac{\sum_{h=1}^{H} \alpha_h \left[\sum_{t=1}^{n-h} y_t y_{t+h} - \rho_h \sum_{t=1}^{n} y_t^2\right]/\sqrt{n}}{\sum_{t=1}^{n} y_t^2/n} \\ &= \frac{\sum_{h=1}^{H} \alpha_h z_n^{(h)}}{c_0} \end{aligned} \tag{2.41}$$

has a limiting normal distribution for any constants $\alpha_1, \dots, \alpha_H$. The generalization of Lemma 1 pertains to $\sum_{h=1}^{H} \alpha_h z_{nk}^{(h)*}$, which is modified by addition and subtraction of a finite number of terms (the number not depending on n) to

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\sum_{r=1}^{k+H} \delta_{rk}^{H} v_t v_{t-r}, \tag{2.42}$$

where

$$\delta_{rk}^{H} = \sum_{h=1}^{H} \alpha_h \delta_{rk}^{(h)}. \tag{2.43}$$

Lemmas 2 and 3 show that

$$\sum_{h=1}^{H} \alpha_h z_n^{(h)} - \sum_{h=1}^{H} \alpha_h z_{nk}^{(h)*} \xrightarrow{P} 0 \tag{2.44}$$

uniformly in n as $k \to \infty$. Then the theorem follows. ∎

3. Proof of Corollary.

Lemma 6. Under the conditions of the theorem

$$\bar{x} \xrightarrow{P} \mu. \tag{3.1}$$

Proof of Lemma 6. We have

$$\begin{aligned} n\mathcal{E}(\bar{x}-\mu)^2 &= \sum_{s=-(n-1)}^{n-1} \left(1 - \frac{|s|}{n}\right) \mathcal{E}(x_t - \mu)(x_{t+s} - \mu) \\ &= \sum_{s=-(n-1)}^{n-1} \left(1 - \frac{|s|}{n}\right) \sum_{i,j=0}^{\infty} \gamma_i \gamma_j \mathcal{E} v_{t-i} v_{t+|s|-j} \\ &= \sum_{s=-(n-1)}^{n-1} \left(1 - \frac{|s|}{n}\right) \sum_{i=0}^{\infty} \gamma_i \gamma_{i+|s|} \mathcal{E} v_{t-i}^2 \\ &\leq \sum_{s=-\infty}^{\infty} \sum_{i=0}^{\infty} |\gamma_i| \left|\gamma_{i+|s|}\right| K \\ &= \left(\sum_{i=0}^{\infty} |\gamma_i|\right)^2 K \\ &< \infty. \end{aligned} \tag{3.2}$$

Then (3.1) follows by Tchebycheff's inequality. ∎

Proof of Corollary. We have

(3.3)

$$
\begin{aligned}
\sqrt{n}[c_h^* - \sigma(h)] &= \frac{1}{\sqrt{n}}\left[\sum_{t=1}^{n-h}(x_t - \bar{x})(x_{t+h} - \bar{x}) - n\sigma(h)\right] \\
&= \frac{1}{\sqrt{n}}\left\{\sum_{t=1}^{n-h}[(x_t - \mu) - (\bar{x} - \mu)][(x_{t+h} - \mu) - (\bar{x} - \mu)] - n\sigma(h)\right\} \\
&= \sqrt{n}[c_h - \sigma(h)] - (\bar{x} - \mu)\frac{1}{\sqrt{n}}\sum_{t=1}^{n-h}(x_t - \mu) \\
&\qquad - (\bar{x} - \mu)\frac{1}{\sqrt{n}}\sum_{t=1}^{n-h}(x_{t+h} - \mu) + \frac{n-h}{\sqrt{n}}(\bar{x} - \mu)^2 \\
&= \sqrt{n}[c_h - \sigma(h)] + o_p(1).
\end{aligned}
$$

Since the set $\sqrt{n}[c_1^* - \sigma(1)], \ldots, \sqrt{n}[c_H^* - \sigma(H)]$ has the same limiting distribution as the set $\sqrt{n}[c_1 - \sigma(1)], \ldots, \sqrt{n}[c_H - \sigma(H)]$ and $c_0^* \xrightarrow{p} \sigma(0)$, the limiting distribution of the set $\sqrt{n}(r_1^* - \rho_1), \ldots, \sqrt{n}(r_H^* - \rho_H)$ is the same as the limiting distribution of the set $\sqrt{n}(r_1 - \rho_1), \ldots, \sqrt{n}(r_H - \rho_H)$. ∎

4. Discussion

4.1. Assumptions on Moments of Innovations. Although the asymptotic distribution of autocovariances may depend on finiteness or boundedness of fourth-order moments, the asymptotic distribution of the autocorrelations has been demonstrated here on the basis of boundedness of second-order moments and of mixed fourth-order moments; these conditions are weaker than the condition of boundedness of the pure fourth-order moments. In fact the boundedness of second-order moments is used in Lemmas 3,4, and 6, and the boundedness of mixed fourth-order moments is applied in Lemma 2; these conditions may be slightly unnecessarily strong. However, (1.20), which involves mixed fourth-order products, is essential to obtain the covariances of the limiting distributions. It should be noted that the

conditions do permit heterogeneity of variances. In the case that the innovations are independently identically distributed only the second- order moment is assumed finite, but that implies that the mixed fourth-order moments are finite and are determined by the second-order moments (and the fact that the first-order moment is 0).

4.2. Comparison with Autoregression. In the case of the process defined by (1.10) the boundedness of second-order moments and mixed fourth-order moments is not needed. Conditions (1.14), (1.15), (1.16), (1.17), and (1.20) suffice. In fact, it is not assumed that $\mathcal{E}v_t^2 < \infty$. See Anderson and Kunitomo (1989).

4.3. A More General Mixed Fourth-Order Moment Limit. Instead of (1.20) we can assume

$$\frac{1}{n}\sum_{t=1}^{n}\sigma_t^2 v_{t-r}v_{t-s} \xrightarrow{\text{p}} \tau_{rs}\sigma^4, \qquad r,s=1,2,\ldots, \tag{4.1}$$

where $\{\tau_{rs}\}$ is arbitrary. Then in the theorem and corollary w_{gh} given by (1.17) is replaced by

$$\sum_{r,s=1}^{\infty}\tau_{rs}(\rho_{r+g}+\rho_{r-g}-2\rho_r\rho_g)(\rho_{r+h}+\rho_{r-h}-2\rho_r\rho_h). \tag{4.2}$$

The major change in the derivation is that a generalization of Theorem 1 of Anderson and Kunitomo (1989) is applied to obtain a variance in Lemma 1 of

$$\sum_{r,s=1}^{k+l}\tau_{rs}\delta_{rk}^{(l)}\delta_{sk}^{(l)}. \tag{4.3}$$

4.4. Comparison with a Theorem of Hannan and Heyde. As noted in the introduction, Hannan and Heyde (1972) generalized the theorem of Anderson and Walker (1964) to innovations being martingale differences. The conditions in

the present paper [as in Anderson and Kunitomo (1989)] are weaker than those of Hannan and Heyde. In both papers (1.6), (1.14), and (1.21) are assumed. Instead of (1.17), (1.18), and (1.19), Hannan and Heyde assume that there exists a constant c and a random variable X such that

$$\Pr\{|v_t| > u\} \leq c \ \Pr\{|X| > u\}, \ \forall u > 0, \tag{4.4}$$

and

$$\mathcal{E}X^4 < \infty. \tag{4.5}$$

This condition implies that the pure fourth-order moments of the innovations are bounded. Instead of (1.16) they assume

$$\frac{1}{n}\sum_{t=1}^{n} \sigma_t^2 \rightarrow \sigma^2 \qquad \text{a.s.} \tag{4.6}$$

and

$$\mathcal{E}v_t^2 = \sigma^2. \tag{4.7}$$

In addition to (4.1) they assume

$$\mathcal{E}v_t^2 v_{t-r} v_{t-s} = \sigma^4 \tau_{rs}, \qquad r, s = 1, 2, \ldots. \tag{4.8}$$

The conditions of Hannan and Heyde are considerably stronger than those of this paper in that they assume that variances and mixed fourth-order moments of the innovations are homogeneous.

References

Anderson, R. L. (1942), Distribution of the serial correlation coefficient, *Ann. Math. Statist.*, **13**, 1-13.

Anderson, R. L., and T. W. Anderson (1950), Distribution of the circular serial correlation coefficient for residuals from a fitted Fourier series, *Ann. Math. Statist.*, **21**, 59-81.

Anderson, T. W. (1948), On the theory of testing serial correlation, *Skand. Aktuarietidskr.*, **31**, 88-116.

Anderson, T. W. (1959), On asymptotic distributions of estimates of parameters of stochastic difference equations, *Ann. Math. Statist.*, **30**, 676-687.

Anderson, T. W., and Naoto Kunitomo (1989), Asymptotic robustness in regression and autoregression based on Lindeberg conditions, Technical Report No. 23, ARO Contract No. DAAL03-89-K-0033, Department of Statistics, Stanford University.

Anderson, T. W., and A. M. Walker (1964), On the asymptotic distribution of the autocorrelations of a sample from a linear stochastic process, *Ann. Math. Statist.*, **35**, 1296-1303.

Bartlett, M. S. (1946), On the theoretical specification and sampling properties of autocorrelated time-series, *J. Roy. Statist. Soc. Supp.* **8**, 27-41, 85-97. [Corrigenda (1948), **10**, 200.]

Diananda, P. H. (1953), Some probability limit theorems with statistical applications, *Proc. Cambridge Philos. Soc.* **49**, 239-246.

Dixon, Wilfrid J. (1944), Further contributions to the problem of serial correlation, *Ann. Math. Statist.*, **15**, 119-144.

Durbin, J., and G. S. Watson (1950), Testing for serial correlation in least squares regression. I, *Biometrika*, **37**, 409-428.

Durbin, J., and G. S. Watson (1951), Testing for serial correlation in least squares regression. II, *Biometrika*, **38**, 159-178.

Dvoretzky, Aryeh (1972), Asymptotic normality for sums of dependent random variables, *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability*, Volume 2. University of California Press, Berkeley and Los Angeles, 513-535.

Hall, P., and C. C. Heyde (1980), *Martingle Limit Theory and Its Application*, Academic Press, New York.

Hannan, E. J., and C. C. Heyde (1972), On limit theorems for quadratic functions of discrete time series, *Ann. Math. Statist.*, **43**, 2058-2066.

Hoeffffding, Wassily, and Herbert Robbins (1948), The central limit theorem for dependent random variables, *Duke Math. J.*, **15**, 773-780.

Koopmans, Tjalling (1942), Serial correlation and quadratic forms in normal variates, *Ann. Math. Statis.*, **13**, 14-33.

Mann, H. B. and A. Wald (1943), On the statistical treatment of linear stochastic difference equations, *Econometrica*, **11**, 173-220.

von Neumann, John (1941), Distribution of the ratio of the mean square successive difference to the variance, *Ann. Math. Statist.*, **12**, 367-395.

Walker, A. M. (1954), The asymptotic distribution of serial correlation coefficients for autoregressive processes with dependent residuals, *Proc. Cambridge Philos. Soc.*, **50**, 60-64.

Watson, G. S. (1952), *Serial Correlation in Regression Analysis*, Mimeo Series No. 49, Institute of Statistics, University of North Carolina, Chapel Hill, N.C.

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503.

1. AGENCY USE ONLY (Leave blank)

2. REPORT DATE
April 1991

3. REPORT TYPE AND DATES COVERED
Technical Report

4. TITLE AND SUBTITLE
The Asymptotic Distribution of Autoregressive Coefficients

5. FUNDING NUMBERS
DAAL03-89-K-0033

6. AUTHOR(S)
T. W. Anderson

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Department of Statistics
Sequoia Hall
Stanford University
Stanford, CA 94305-4065

8. PERFORMING ORGANIZATION REPORT NUMBER
26

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
U. S. Army Research Office
P. O. Box 12211
Research Triangle Park, NC 27709-2211

10. SPONSORING/MONITORING AGENCY REPORT NUMBER
ARO 26394.5-MA

11. SUPPLEMENTARY NOTES
The view, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation.

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*

The asymptotic distribution of a finite set of autocorrelations is obtained for a time series from a linear stochastic process. The disturbances (or innovations) are martingale differences with bounded variances and bounded mixed fourth-order moments satisfying a uniform conditional square integrability condition. The conditions are weaker than those used previously for such asymptotic distributions.

14. SUBJECT TERMS
autocorrelations, asymptotic distributions, martingale differences, bounded second-order moments

15. NUMBER OF PAGES
21

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	UL